JOURNAL OF APPROXIMATION THEORY 43, 383–393 (1985)

A Quantitative Condensation of Singularities on Arbitrary Sets

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Communicated by P. Nevai

Received April 30, 1984

This paper deals with quantitative extensions of the classical condensation principle of Banach and Steinhaus to arbitrary (not necessarily countable) families of sequences of operators. Some applications concerned with the sharpness of approximation processes, with (Weierstrass) continuous nondifferentiable functions as well as with the classical counterexample of Marcinkiewicz on the divergence of Lagrange interpolation polynomials, illustrate this unifying approach to various condensations of singularities in analysis.

1. INTRODUCTION

This paper is concerned with condensation of singularities on arbitrary sets. Continuing our previous investigations in [5], we derive a general quantitative condensation principle, followed by some illustrative applications to approximation theory. To motivate the abstract results, let us recall the classical condensation principle which is concerned with double sequences of operators.

With $\mathbb{C}$ and $\mathbb{N}$, the set of complex and natural numbers, respectively, let X be a complex Banach space (with norm $\|\circ\|$) and X^* be the class of functionals T on X which are sublinear, i.e.,

$$|T(f+g)| \leqslant |Tf| + |Tg|, \qquad |T(af)| = |a|\,|Tf|$$

for all $f, g \in X$ and $a \in \mathbb{C}$, and which are bounded, i.e.,

$$\|T\|_{X^*} := \sup\{|Tf|;\ \|f\| = 1\} < \infty. \tag{1.1}$$

Given a double sequence $\{T_{n,\lambda}\} \subset X^*$ (e.g., remainder of some pointwise approximation process, taken at a denumerable set of points), the classical condensation principle states (see [1]):

0021-9045/85 $3.00

CONDENSATION PRINCIPLE (BANACH–STEINHAUS 1927). If for $\{\{T_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\mathbb{N}\}\subset X^*$ and each $\lambda\in\mathbb{N}$

$$\limsup_{n\to\infty}\|T_{n,\lambda}\|_{X^*}=\infty,$$

then there exists $f_0\in X$, independent of n, $\lambda\in\mathbb{N}$, such that

$$\limsup_{n\to\infty}|T_{n,\lambda}f_0|=\infty \tag{1.2}$$

simultaneously for each $\lambda\in\mathbb{N}$.

For our purposes it is appropriate to point out the following equivalent (cf. (1.1)) formulation, given in [7, p. 20 ff.]:

CONDENSATION PRINCIPLE (KACZMARZ–STEINHAUS 1935). If for $\{\{T_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\mathbb{N}\}\subset X^*$ and each n, $\lambda\in\mathbb{N}$ there exists $g_{n,\lambda}\in X$ satisfying

$$\|g_{n,\lambda}\|\leqslant C_1, \tag{1.3}$$

$$\limsup_{n\to\infty}|T_{n,\lambda}\,g_{n,\lambda}|=\infty \tag{1.4}$$

for each $\lambda\in\mathbb{N}$, then there exists $f_0\in X$ for which (1.2) holds true.

In [4] we introduced the following quantitative extensions, still for denumerable sets of λ: Let $U\subset X$ be a seminormed linear subspace with seminorm $|\circ|_U$. Consider the Peetre K-functional $(t\geqslant 0)$

$$K(t,f;X,U):=\inf\{\|f-g\|+t\,|g|_U;\,g\in U\},$$

a standard measure of smoothness for elements of abstract spaces. Then intermediate spaces (abstract Lipschitz classes) are defined by

$$X_\omega:=\{f\in X;\,K(t,f;X,U)=\mathcal{O}_f(\omega(t)),\ t\to 0+\},$$

where ω is a function, continuous on $(0,\infty)$ such that

$$0<\omega(s)\leqslant\omega(s+t)\leqslant\omega(s)+\omega(t)\qquad(0<s,\,t) \tag{1.5}$$

(abstract modulus of continuity, cf. [11, p. 96 ff.]). It follows that $\liminf_{t\to 0+}\omega(t)/t>0$, in fact

$$\omega(t)/t\leqslant 2\omega(s)/s\qquad(0<s\leqslant t). \tag{1.6}$$

Often ω is assumed to satisfy additionally

$$\lim_{t\to 0+}\omega(t)=0,\qquad \lim_{t\to 0+}\omega(t)/t=\infty. \tag{1.7}$$

Let $\{\{\varphi_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\mathbb{N}\}$ be a double sequence of (strictly) positive numbers with $\lim_{n\to\infty}\varphi_{n,\lambda}=0$ for each $\lambda\in\mathbb{N}$. In these terms one has (see [4]).

CONDENSATION PRINCIPLE (WITH LARGE $\mathcal{O}$-RATES). If for $\{\{T_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\mathbb{N}\}\subset X^*$ and each $n,\lambda\in\mathbb{N}$ there exists $g_{n,\lambda}\in U$ satisfying (1.3, 4) and

$$|g_{n,\lambda}|_U\leqslant C_2/\varphi_{n,\lambda}, \tag{1.8}$$

then for each ω with (1.5) there exists $f_\omega\in X_\omega$, independent of $n,\lambda\in\mathbb{N}$, such that

$$\limsup_{n\to\infty}|T_{n,\lambda}f_\omega|/\omega(\varphi_{n,\lambda})=\infty \tag{1.9}$$

simultaneously for each $\lambda\in\mathbb{N}$.

Dealing with (proper) rates, one may even replace large $\mathcal{O}$-rates by small o-ones in the following sense (see [4]):

CONDENSATION PRINCIPLE (WITH SMALL o-RATES). If for $\{\{T_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\mathbb{N}\}\subset X^*$ and each $n,\lambda\in\mathbb{N}$ there exists $g_{n,\lambda}\in U$ satisfying (1.3, 8) and (instead of (1.4))

$$\limsup_{n\to\infty}|T_{n,\lambda}\,g_{n,\lambda}|\geqslant C_3>0, \tag{1.10}$$

then for each ω with (1.5, 7) there exists $f_\omega\in X_\omega$, independent of $n,\lambda\in\mathbb{N}$, such that

$$\limsup_{n\to\infty}|T_{n,\lambda}f_\omega|/\omega(\varphi_{n,\lambda})\geqslant C_3 \tag{1.11}$$

simultaneously for each $\lambda\in\mathbb{N}$.

It is the latter aspect we would like to extend to the case that the parameter λ varies over an arbitrary set Λ (instead of $\mathbb{N}$). However, whereas the quantitative extensions in case of denumerable parameter sets could be developed quite naturally from the classical principle, there are some further, rather technical conditions entering the picture when dealing with arbitrary sets Λ. In fact, we have to replace the (natural) lim sup in condition (1.10) by the corresponding lim inf (cf. (2.2)). On the other hand, instead of the K-functional, it is often more convenient for the applications to employ other appropriate functionals $S_t\in X^*$, $t\to 0+$, as a measure of smoothness. Accordingly, a rather general theorem is established in Section 2, which meets the needs in many cases of interest. This is illustrated in Section 3, where some first applications are worked out, indicating the usefulness of the present abstract approach.

2. A Quantitative Condensation Principle

In the following Λ denotes an arbitrary set. Moreover, $\{\varphi_n\}_{n\in\mathbb{N}}$ will always be a sequence of (strictly) positive numbers with $\lim_{n\to\infty}\varphi_n=0$, and $\sigma(t)$ a (strictly) positive function on $(0,\infty)$.

THEOREM 2.1. *Suppose that for $\{\{T_{n,\lambda}\}_{n\in\mathbb{N}};\lambda\in\Lambda\}\subset X^*$ and each $n\in\mathbb{N}$ there exists $g_n\in X$ such that (cf. (1.3, 10))*

$$\|g_n\|\leqslant C_1, \tag{2.1}$$

$$\liminf_{n\to\infty}|T_{n,\lambda}g_n|\geqslant C_{3,\lambda}>0 \tag{2.2}$$

for each $\lambda\in\Lambda$, and additionally

$$\|T_{n,\lambda}\|_{X^*}\leqslant C_{4,n}\qquad(\lambda\in\Lambda), \tag{2.3}$$

$$|T_{n,\lambda}g_j|\leqslant C_{5,\lambda}C_{6,j}\varphi_n\qquad(1\leqslant j\leqslant n-1,\lambda\in\Lambda). \tag{2.4}$$

Furthermore, let $\{S_t;t\in(0,\infty)\}\subset X^*$ be a family of functionals satisfying

$$|S_tg_n|\leqslant C_7\min\{1,\sigma(t)/\varphi_n\}\qquad(n\in\mathbb{N},t>0). \tag{2.5}$$

Then for each ω with (1.5, 7) there exists $f_\omega\in X$, independent of $n\in\mathbb{N}$, $\lambda\in\Lambda$, such that

$$|S_tf_\omega|\leqslant 6C_7\omega(\sigma(t))\qquad(t>0) \tag{2.6}$$

(instead of $f_\omega\in X_\omega$) and (cf. (1.11))

$$\limsup_{n\to\infty}|T_{n,\lambda}f_\omega|/\omega(\varphi_n)\geqslant\tfrac{1}{2}C_{3,\lambda} \tag{2.7}$$

simultaneously for each $\lambda\in\Lambda$.

Proof. The method of proof consists in a suitable quantitative extension of the familiar gliding hump method (cf. [5, 6]). Starting with $n_1=1$, one may successively construct a subsequence $\{n_k\}\subset\mathbb{N}$ such that for $k\geqslant 2$ (cf. (1.7))

$$n_k>n_{k-1},\qquad\varphi_{n_k}<\varphi_{n_{k-1}}, \tag{2.8}$$

$$\sum_{j=1}^{k-1}\omega(\varphi_{n_j})\max\{C_{6,n_j},1/\varphi_{n_j}\}\leqslant\omega(\varphi_{n_k})/k\varphi_{n_k}, \tag{2.9}$$

$$\omega(\varphi_{n_k})\leqslant\min\{1,1/C_{4,n_{k-1}}\}\,\omega(\varphi_{n_{k-1}})/(k-1). \tag{2.10}$$

Since X is complete and (cf. (2.1, 10))

$$\sum_{j=k+1}^{\infty} \omega(\varphi_{n_j}) \, \|g_{n_j}\| \leqslant C_1 \omega(\varphi_{n_{k+1}}) \sum_{j=0}^{\infty} (k+1)^{-j} \leqslant 2C_1 \omega(\varphi_{n_{k+1}}), \quad (2.11)$$

the element

$$f_\omega := \sum_{j=1}^{\infty} \omega(\varphi_{n_j}) \, g_{n_j}$$

is well defined in X. Suppose that $t \in (0, \infty)$ is such that $\sigma(t) \leqslant \varphi_1$. Then there exists $k \in \mathbb{N}$ such that $\varphi_{n_{k+1}} < \sigma(t) \leqslant \varphi_{n_k}$ (cf. (2.8)), and it follows by (1.5, 6), (2.5, 9, 11) that

$$\begin{aligned} |S_t f_\omega| &\leqslant \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty} \right) \omega(\varphi_{n_j}) \, |S_t \, g_{n_j}| \\ &\leqslant C_7 \sigma(t) \sum_{j=1}^{k} \omega(\varphi_{n_j})/\varphi_{n_j} + C_7 \sum_{j=k+1}^{\infty} \omega(\varphi_{n_j}) \\ &\leqslant C_7 \sigma(t)(1+k^{-1}) \, \omega(\varphi_{n_k})/\varphi_{n_k} + 2C_7 \omega(\varphi_{n_{k+1}}) \\ &\leqslant 2C_7(2\omega(\sigma(t)) + \omega(\sigma(t))) = 6C_7 \omega(\sigma(t)). \end{aligned}$$

If $t \in (0, \infty)$ is such that $\sigma(t) > \varphi_1$, then by (1.5), (2.5)

$$|S_t f_\omega| \leqslant C_7 \sum_{j=1}^{\infty} \omega(\varphi_{n_j}) \leqslant C_7 2\omega(\varphi_1) \leqslant 2C_7 \omega(\sigma(t)),$$

thus in any case (2.6). Now, given $\lambda \in \Lambda$, there exists $m_\lambda \in \mathbb{N}$ such that $|T_{n,\lambda} \, g_n| \geqslant C_{3,\lambda}/2$ for all $n \geqslant m_\lambda$ (cf. (2.2)). Hence for all $n_k \geqslant m_\lambda$ (cf. (2.3), (2.4), (2.9)–(2.11))

$$\begin{aligned} |T_{n_k,\lambda} f_\omega| &\geqslant \omega(\varphi_{n_k}) \, |T_{n_k,\lambda} \, g_{n_k}| - \left(\sum_{j=1}^{k-1} + \sum_{j=k+1}^{\infty} \right) \omega(\varphi_{n_j}) \, |T_{n_k,\lambda} \, g_{n_j}| \\ &\geqslant \omega(\varphi_{n_k}) \, C_{3,\lambda}/2 - \sum_{j=1}^{k-1} \omega(\varphi_{n_j}) \, C_{5,\lambda} C_{6,n_j} \varphi_{n_k} \\ &\quad - 2C_1 C_{4,n_k} \omega(\varphi_{n_{k+1}}) \\ &\geqslant \omega(\varphi_{n_k}) \left[\frac{1}{2} C_{3,\lambda} - \frac{1}{k} C_{5,\lambda} - \frac{2}{k} C_1 \right]. \end{aligned}$$

Therefore for each $\lambda \in \Lambda$

$$\limsup_{k \to \infty} |T_{n_k,\lambda} f_\omega| / \omega(\varphi_{n_k}) \geqslant \tfrac{1}{2} C_{3,\lambda},$$

thus (2.7). This completes the proof. ∎

Note that for $S_t f = K(t, f; X, U)$ condition (2.5) may be replaced by the Bernstein-type inequality (cf. (1.8))

$$g_n \in U \qquad \text{and} \qquad |g_n|_U \leqslant C_2/\varphi_n \qquad (n \in \mathbb{N}).$$

Indeed, this implies (2.5) with $C_7 = \max\{C_1, C_2\}$ and $\sigma(t) = t$ since

$$\begin{aligned} K(t, f; X, U) &\leqslant \|f\| && (f \in X) \\ &\leqslant t\,|f|_U && (f \in U). \end{aligned}$$

Thus Theorem 2.1 subsumes the condensation principle of [5] as well as, in case of a one-point parameter set Λ, the uniform boundedness principle with small o-rates, given in [3].

Moreover, Theorem 2.1 contains the following condensation principle (without rates) for arbitrary parameter sets Λ, which may be compared with the classical one for double sequences, referred to in Section 1.

COROLLARY 2.2. *Suppose that for* $\{\{T_{n,\lambda}\}_{n \in \mathbb{N}}; \lambda \in \Lambda\} \subset X^*$ *and each* $n \in \mathbb{N}$ *there exists* $g_n \in X$ *such that* (2.1) *and* (*instead of* (2.2))

$$\liminf_{n \to \infty} |T_{n,\lambda} g_n|/C_{3,n} \geqslant 1 \qquad \text{with} \quad \limsup_{n \to \infty} C_{3,n} = \infty \tag{2.12}$$

are satisfied for each $\lambda \in \Lambda$, *and additionally* (2.3) *and* (*instead of* (2.4))

$$|T_{n,\lambda} g_j| \leqslant C_{5,\lambda} C_{6,j} \qquad (1 \leqslant j \leqslant n-1,\ \lambda \in \Lambda). \tag{2.13}$$

Then there exists $f_0 \in X$, *independent of* $n \in \mathbb{N}$, $\lambda \in \Lambda$, *such that*

$$\limsup_{n \to \infty} |T_{n,\lambda} f_0| = \infty \tag{2.14}$$

simultaneously for each $\lambda \in \Lambda$.

Proof. In view of (2.12, 13) one may select a strictly increasing subsequence $\{n_k\} \subset \mathbb{N}$ with $C_{3,n_k} \geqslant k^2$ such that the functionals $\tilde{T}_{k,\lambda} := k^{-2} T_{n_k,\lambda}$ satisfy

$$\liminf_{k \to \infty} |\tilde{T}_{k,\lambda} g_{n_k}| \geqslant 1,$$

$$|\tilde{T}_{k,\lambda} g_{n_j}| \leqslant C_{5,\lambda} C_{6,n_j} k^{-2} \qquad (1 \leqslant j \leqslant k-1).$$

Thus (2.1)–(2.4) hold true for $\tilde{T}_{k,\lambda}$, $\tilde{g}_k := g_{n_k}$ with $C_{3,\lambda} = 1$ and $\varphi_k = k^{-2}$. An application of Theorem 2.1 for $\omega(t) = t^{1/2}$ (and $S_t = 0$) gives the existence of an element $f_0 \in X$ satisfying (2.7), i.e., for each $\lambda \in \Lambda$

$$\tfrac{1}{2} \leqslant \limsup_{k \to \infty} |\tilde{T}_{k,\lambda} f_0|\, k = \limsup_{k \to \infty} |T_{n_k,\lambda} f_0|/k.$$

This immediately implies (2.14). ∎

As is indicated in Section 3.4, the condensation principle of Corollary 2.2 admits a unified approach in regard to the existence of some classical counterexamples in approximation theory.

3. APPLICATIONS

In the following $C_{2\pi}$, $C[a, b]$, and $C_0(\mathbb{R})$ denote the spaces of complex-valued functions f, continuous on the real axis $\mathbb{R}$ with period 2π, on the compact interval $[a, b] \subset \mathbb{R}$, and on $\mathbb{R}$ with $\lim_{|x| \to \infty} f(x) = 0$, respectively, each one endowed with the corresponding maximum norm $\|f\|_C$. As measures of smoothness we consider usual moduli of continuity, thus

$$\omega_1(t, f; C_{2\pi}) := \sup_{|h| \leqslant t} \max_{x \in \mathbb{R}} |f(x+h) - f(x)|,$$

$$\omega_1(t, f; C[a, b]) := \sup_{0 \leqslant h \leqslant t} \max_{a \leqslant x \leqslant b-h} |f(x+h) - f(x)|,$$

$$\omega_2(t, f; C_0(\mathbb{R})) := \sup_{|h| \leqslant t} \max_{x \in \mathbb{R}} |f(x+h) - 2f(x) + f(x-h)|.$$

Corresponding Lipschitz classes are then given for $\alpha > 0$ by, e.g.,

$$\mathrm{Lip}_2(\alpha; C_0(\mathbb{R})) := \{f \in C_0(\mathbb{R}); \omega_2(t, f; C_0(\mathbb{R})) = \mathcal{O}_f(t^\alpha), t \to 0+\}.$$

3.1. *Fejér Means*

Let us start with the classical Fejér means ($n \in \mathbb{N}$, $x \in \mathbb{R}$)

$$F_{n-1}(f; x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) f^{\wedge}(k)\, e^{ikx}, \qquad f^{\wedge}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, e^{-iku}\, du,$$

of the Fourier series of $f \in C_{2\pi}$. It is well known that for each $f \in \mathrm{Lip}_1(\alpha; C_{2\pi})$, $0 < \alpha < 1$,

$$|F_{n-1}(f; x) - f(x)| = \mathcal{O}_f(n^{-\alpha}) \qquad (n \to \infty)$$

uniformly for all $x \in \mathbb{R}$. An application of Theorem 2.1 yields the sharpness of this approximation assertion in the following pointwise sense:

COROLLARY 3.1. *For each* $\alpha \in (0, 1)$ *there exists* $f_\alpha \in \mathrm{Lip}_1(\alpha; C_{2\pi})$ *such that simultaneously for* each $x \in \mathbb{R}$

$$\limsup_{n \to \infty} n^\alpha |F_{n-1}(f_\alpha; x) - f_\alpha(x)| \geqslant 1.$$

Proof. For $X = C_{2\pi}$, $\Lambda = \mathbb{R}$, $g_n(x) = e^{inx}$, $T_{n,x}f = |F_{n-1}(f; x) - f(x)|$, and $S_t f = \omega_1(t, f; C_{2\pi})$ the conditions of Theorem 2.1 are satisfied with $C_1 = 1$, $C_{3,x} = 1$, $C_{4,n} = 2$, $C_{5,x} = 1$, $C_{6,j} = j$, $\varphi_n = 1/n$, $C_7 = 2$, and $\sigma(t) = t$. This is a consequence of

$$|F_{n-1}(g_j; x) - g_j(x)| = \min\{1, j/n\} \; |e^{ijx}|,$$

$$|F_{n-1}(f; x) - f(x)| \leqslant 2 \; \|f\|_C,$$

$$\omega_1(t, g_n; C_{2\pi}) = \sup_{|h| \leqslant t} |e^{inh} - 1| \; \|e^{inx}\|_C \leqslant \min\{2, nt\}.$$

Thus for $\omega(t) = t^\alpha$, $0 < \alpha < 1$, one obtains a function $f_\alpha \in C_{2\pi}$ with (2.6, 7), which completes the proof. ∎

In quite the same way one may treat any other of the classical summation processes of Fourier series.

3.2. *Singular Integral of Gauss and Weierstrass*

The situation concerning the construction of test elements g_n changes slightly in the Fourier integral case on, e.g., the space $C_0(\mathbb{R})$. To this end, let us consider the singular integral of Gauss and Weierstrass

$$W_t(f; x) := (4\pi t)^{-1/2} \int_{\mathbb{R}} f(x-u) \, e^{-u^2/4t} \, du \qquad (t > 0, x \in \mathbb{R}).$$

For $f \in \mathrm{Lip}_2(\alpha; C_0(\mathbb{R}))$, $0 < \alpha \leqslant 2$, one has the familiar estimate

$$\|W_t(f; x) - f(x)\|_C = \mathcal{O}_f(t^{\alpha/2}) \qquad (t \to 0+).$$

Again Theorem 2.1 delivers the pointwise sharpness of this result for the class $\mathrm{Lip}_2(\alpha; C_0(\mathbb{R}))$.

COROLLARY 3.2. *For each* $\alpha \in (0, 2)$ *there exists* $f_\alpha \in \mathrm{Lip}_2(\alpha; C_0(\mathbb{R}))$ *such that simultaneously for* each $x \in \mathbb{R}$

$$\limsup_{t \to 0+} t^{-\alpha/2} \, |W_t(f_\alpha; x) - f_\alpha(x)| \geqslant 1.$$

Proof. Set $X = C_0(\mathbb{R})$, $\Lambda = \mathbb{R}$, $T_{n,x}f = |W_{1/n}(f; x) - f(x)|$, $S_t f = \omega_2(t, f; C_0(\mathbb{R}))$, and $g_n(x) = H(x/n) \exp\{ixn^{1/2}\}$, where H is an arbitrarily often differentiable function with compact support such that $H(x) = 1$ for $|x| \leqslant 1$. Thus, with $\tilde{g}_n(x) := \exp\{ixn^{1/2}\}$, $H_n(x) = H(x/n)$ (hence $g_n = H_n \tilde{g}_n$), one obtains for $|x| \leqslant n - 1$

$$\begin{aligned}
&|W_{1/n}(g_n; x) - g_n(x)| \\
&\quad \geqslant |W_{1/n}(\tilde{g}_n; x) - \tilde{g}_n(x)| - |W_{1/n}(\tilde{g}_n(H_n - 1); x)| \\
&\quad \geqslant |\tilde{g}_n(x)|\ |e^{-1} - 1| - (n/4\pi)^{1/2} \int_{\mathbb{R}} |H((x-u)/n) - 1 \mid e^{-u^2 n/4}\, du \\
&\quad \geqslant 1 - 1/e - (\|H\|_C + 1)(n/4\pi)^{1/2} \int_{|u| \geqslant 1} e^{-u^2 n/4}\, du \\
&\quad = 1 - 1/e + o(1) \qquad (n \to \infty).
\end{aligned}$$

Therefore conditions (2.1, 2) are satisfied with $C_1 = \|H\|_C$ and $C_{3,x} = 1 - 1/e$, $x \in \mathbb{R}$. Concerning (2.3)–(2.5), one has (cf. [2, p. 143, 137])

$$|S_t\, g_j| \leqslant t^2\, \|g_j''\|_C, \qquad |T_{n,x}\, g_j| \leqslant (1/n)\, \|g_j''\|_C$$

as well as $\|g_j''\|_C \leqslant Mj$ so that the conditions in question are satisfied with $C_{4,n} = 2$, $C_{5,x} = M$, $C_{6,j} = j$, $\varphi_n = 1/n$, $C_7 = \max\{4\ \|H\|_C, M\}$, and $\sigma(t) = t^2$. Hence (2.6, 7) for $\omega(t) = t^{\alpha/2}$, $0 < \alpha < 2$, deliver the assertion. ∎

3.3. *Nowhere Differentiable Functions*

The following application, concerned with functions defined on a compact interval, leads back to the origins in the development of condensation principles and their method of proof, the gliding hump method. For $X = C[a, b]$ consider the functionals $S_t f = \omega_1(t, f; C[a, b])$ and

$$T_{n,x} f = \begin{cases} |f(x + \pi/2n) - f(x)| & \text{for } a \leqslant x \leqslant b - \pi/2n \\ 0 & \text{for } b - \pi/2n < x < b. \end{cases}$$

Since for $g_j(x) = \exp\{ijx\}$

$$\begin{aligned}
|g_j(x+t) - g_j(x)| = |e^{ijt} - 1|\ |g_j(x)| &\leqslant \min\{2, tj\} \\
&= \sqrt{2} \quad \text{for } t = \pi/2j,
\end{aligned}$$

the conditions of Theorem 2.1 are satisfied with $C_1 = 1$, $C_{3,x} = \sqrt{2}$, $x \in [a, b) = A$, $C_{4,n} = 2$, $C_{5,x} = \pi/2$, $C_{6,j} = j$, $\varphi_n = 1/n$, $C_7 = 2$, and $\sigma(t) = t$. Hence

COROLLARY 3.3. *For each* $\alpha \in (0, 1)$ *there exists* $f_\alpha \in \mathrm{Lip}_1(\alpha; C[a, b])$ *such that simultaneously for* each $x \in [a, b)$

$$\limsup_{t \to 0+} t^{-\alpha}\, |f_\alpha(x+t) - f_\alpha(x)| \geqslant 1.$$

Thus there exist functions f_α such that the (uniform) Lipschitz condition $|f(x+t) - f(x)| = \mathcal{O}_f(t^\alpha)$ is everywhere sharp. Of course, such elements f_α

are in particular (Weierstrass) nowhere differentiable functions which are not only continuous, but in fact elements of the smoothness class $\mathrm{Lip}_1(\alpha; C[a, b])$.

3.4. *Lagrange Interpolation*

To consider an application of Corollary 2.2, let $\{x_{j,n}; 1 \leqslant j \leqslant n, n \in \mathbb{N}\}$ denote a triangular matrix of knots $-1 \leqslant x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} \leqslant 1$ and, for $f \in C[-1, 1]$,

$$L_n(f; x) := \sum_{k=1}^{n} f(x_{k,n})\, l_{k,n}(x),\quad l_{k,n}(x) := \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{x - x_{j,n}}{x_{k,n} - x_{j,n}},$$

be the corresponding Lagrange interpolation polynomial of degree $n-1$. For the particular case $x_{j,n}^T = \cos((2j-1)\pi/2n)$ of Tschebyscheff knots, Marcinkiewicz [9] (see also [10, pp. 379–388]) showed

COROLLARY 3.4. *There exists a function* $f_0 \in C[-1, 1]$ *such that simultaneously for* each $x \in (-1, 1)$

$$\limsup_{n \to \infty} |L_n^T(f_0; x)| = \infty. \tag{3.1}$$

To give a proof via Corollary 2.2 let us first quote (cf. [10, pp. 382–385]).

LEMMA 3.5. *For each* $n \in \mathbb{N}$, $n \geqslant 3$, *there is an algebraic polynomial* r_n *and an integer* $N_n \in \mathbb{N}$ *such that* $\|r_n\|_C \leqslant 2$, *and for each* $x \in [-\cos(\pi/n), \cos(\pi/n)]$ *there is a natural* $m(x) \in (n, N_n]$ *such that*

$$|L_{m(x)}^T(r_n; x)| > n. \tag{3.2}$$

Proof of Corollary 3.4. Consider $X = C[-1, 1]$, $A = (-1, 1)$, $g_n = r_n$,

$$T_{n,x} f = \max_{n < m \leqslant N_n} |L_m^T(f; x)|.$$

Then (2.1, 3) are satisfied with $C_1 = 2$, $C_{4,n} = \max_{n < m \leqslant N_n} \max_{-1 \leqslant x \leqslant 1} \sum_{k=1}^{m} |l_{k,m}^T(x)|$, and (2.12) with $C_{3,n} = n$ (cf. (3.2)), whereas (2.13) is a consequence of the interpolation property $L_n^T(r_j; x) = r_j(x)$, valid for sufficiently large n since r_j is known to be a polynomial. Thus assertion (2.14) holds true for some $f_0 \in C[-1, 1]$, which finally implies (3.1). ∎

Let us conclude with the remark that the example of Kolmogorov [8] of an absolutely integrable function with everywhere divergent Fourier series is also based on a lemma, analogous to Lemma 3.5 (see [12, pp. 310–314]), and thus may be subsumed under the frame of the present general approach.

ACKNOWLEDGMENTS

The authors would like to express their sincere gratitude to Erich van Wickeren for a critical reading of the manuscript and many valuable suggestions. Work on the present subject is supported in part by DFG, Grant Ne 171/5-1, which is also gratefully acknowledged.

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